

VIRTUAL MASS OF REGULAR POLYGON AT ANGLE OF ATTACK

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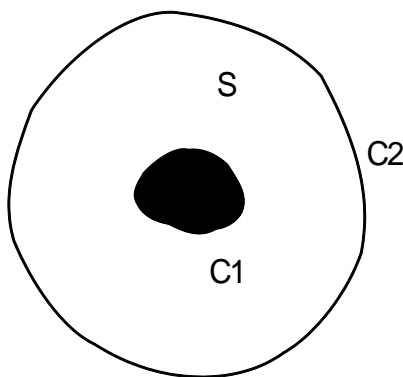
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Graphical abstract



Abstract

In case of the accelerating motion like a sinusoidal motion, the virtual mass effect should be taken into account in equation of motion. Generally speaking we should find the appropriate mapping functions while they are restricted in some functions. To the author's knowledge, the virtual mass of the regular polygon is not shown by the exact solution, while the regular polygon is used as a sectional shape of the structure like a building, a membrane of a structure and so on. It is shown that the virtual mass of an regular polygon has been calculated by using the exact conformal mapping, in which the angle of attack is taken into account. Results show that it is not dependent from the angle of attack except the flat plate ($n=2$). It means, although the body shape is not a point symmetry, there is no dependence of angle of attack on virtual mass except the flat plate.

Keywords: Virtual mass, Regular polygon, Conformal mapping, Angle of attack

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1.0 INTRODUCTION

When the body is accelerated by fluid for instance, the body motion is expressed by the equation of motion like $m\ddot{x} + c\dot{x} + kx = f(t)$. Here m (mass) is composed by both the pure mass of body and the 'virtual' mass. Especially, when the acceleration is large or the mass ratio is almost larger than unity, the effect should not be ignored. Although the virtual mass effect is very important term in motion of equation, a little restricted cases could have been investigated because of the difficulty of calculating it analytically and/or exactly. While the restricted cases are gotten from the famous text (Lamb [1]), Milne-Thomson [2], Landau and Lifshitz [3]), the body geometries are restricted in the circle, ellipse, etc. Here we need to find the conformal mapping function from the geometry (z -plane) to the circle (ζ -plane), while it is very difficult to find a general function for the conformal mapping.

In this article the object is to show the virtual mass of the regular polygon which geometry is used in architecture sometimes, that is not shown at present.

The conformal mapping function is given by the infinite series like $\sum_{n=1}^{\infty} \frac{a_n}{z^n}$.

It is shown that the virtual mass of regular polygon is independent of the angle of attack although the body geometry is not a point symmetry except $n=2$ (flat plate with angle of attack). Finally calculated special examples are shown.

2.0 ANALYSIS

2.1 Virtual Mass and Kinetic Energy

The flow field around a body at angle of attack ' α ' is expressed by using complex velocity potential ' f ' as follows,

$$f = Uz e^{-i\alpha} + w \quad (1)$$

where U : speed of uniform flow, w : 'disturbance' from 'body'.

For example, the flow around a circle is expressed

$$f = U \left(ze^{-i\alpha} + \frac{a^2}{z} e^{i\alpha} \right) \quad (2)$$

$$= Uze^{-i\alpha} + w \quad (3)$$

$$= \text{'Uniform flow'} + \text{'Disturbance'} \quad (4)$$

After Milne-Thomson, the total kinetic energy is derived from 'disturbance' as follows,

$$T = \frac{1}{2} \rho \oint_s \frac{\partial}{\partial z} \left(w \frac{d\bar{w}}{dz} \right) dS \quad (5)$$

This equation is reduced to the following equation (6) using 'Area Rule'

$$T = -\frac{1}{4} i \rho \oint_{c1} w d\bar{w} + \frac{1}{4} i \rho \oint_{c2} w d\bar{w},$$

where $c1$: inner boundary, $c2$: outer boundary. (6)

When $c2$ tends to infinity, the second term is vanished, then

$$T = -\frac{1}{4} i \rho \oint_c w d\bar{w} = \frac{1}{2} U^2 M' = \text{'Total Kinetic Energy'},$$

where M' : virtual mass. (7)

We can determine the virtual mass by using equation (7). Figure 1 shows the graphical of flow field S , inner boundary $C1$ and outer boundary $C2$.

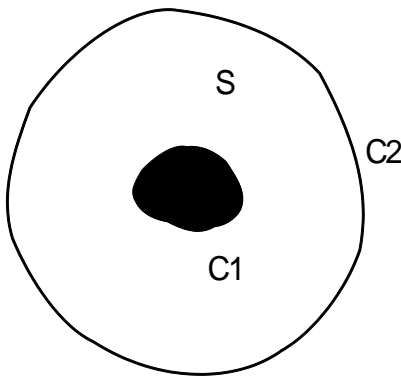


Figure 1 Flow field S , inner boundary $C1$, and outer boundary $C2$

2.2 Mapping Function (circle \Leftrightarrow polygon)

The outer region of the regular polygon with 'n' members is mapped on the outer region of the circle of radius 'a', which conformal mapping function is given in following equation (8) (Imai [4]).

$$\begin{aligned} dz &= \left(1 - \frac{a^n}{\zeta^n} \right)^{\frac{2}{n}} d\zeta \\ &= \left\{ 1 - \frac{2}{n} \left(\frac{a}{\zeta} \right)^n + \frac{\frac{2}{n}(\frac{2}{n}-1)}{2!} \left(\frac{a}{\zeta} \right)^{2n} \right. \\ &\quad \left. - \frac{\frac{2}{n}(\frac{2}{n}-1)(\frac{2}{n}-2)}{3!} \left(\frac{a}{\zeta} \right)^{3n} + \dots \right\} d\zeta \end{aligned} \quad (8)$$

Eq. (8) may be integrated easily to be expressed in the following equation (9).

$$\begin{aligned} z &= \zeta + \underbrace{\frac{2}{n(n-1)} \frac{a^n}{\zeta^{n-1}} - \frac{\frac{2}{n}(\frac{2}{n}-1)}{2!(2n-1)} \frac{a^{2n}}{\zeta^{2n-1}} + \frac{\frac{2}{n}(\frac{2}{n}-1)(\frac{2}{n}-2)}{3!(3n-1)} \frac{a^{3n}}{\zeta^{3n-1}} + \dots}_{A(\zeta)} \\ &= \zeta + \sum_{l=1}^{\infty} \frac{(-1)^{l+1}}{l!(ln-1)} \left\{ \prod_{m=1}^l \left[\frac{2}{n} - (m-1) \right] \right\} \frac{a^{ln}}{\zeta^{ln-1}} \end{aligned} \quad (9)$$

Eq. (9) means that the point A on the z -plane is mapped on the point A' on the ζ -plane, and so on. Here the regular polygon on the z -plane is inscribed in a circle such that the radius is as shown in Figure 2.

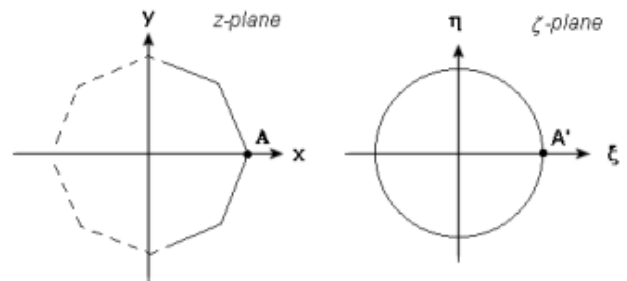


Figure 2 Mapping from the z -plane to the ζ -plane

$$A \left(a + \frac{2}{n(n-1)} a + \dots, 0 \right) \Leftrightarrow A'(a, 0).$$

By the way, the complex velocity potential on the ζ -plane, which represents a flow around a circle, 'a' in radius, with angle of attack, α , is expressed as follows.

$$f = U \left(\zeta e^{-i\alpha} + \frac{a^2}{\zeta} e^{i\alpha} \right) \quad (10)$$

So we can get the complex velocity potential with respect to the 'disturbance' for any geometrical shape in the following equation (11), in which that leads to a 'virtual mass' of any shape of body.

$$\begin{aligned} w &= f - Uze^{-i\alpha} \\ &= U \left[\frac{a^2}{\zeta} e^{i\alpha} - e^{-i\alpha} \{A(\zeta)\} \right] \end{aligned} \quad (11)$$

After taking the total differential and the conjugation of Eq. (11), we get the next Eq. (12).

$$d\bar{w} = U \left[-\frac{a^2}{\zeta^2} e^{-i\alpha} - e^{i\alpha} \underbrace{\left\{ \frac{2}{n} \frac{a^n}{\zeta^n} + \frac{\frac{2}{n}(\frac{2}{n}-1)}{2!} \frac{a^{2n}}{\zeta^{2n}} - \frac{\frac{2}{n}(\frac{2}{n}-1)(\frac{2}{n}-2)}{3!} \frac{a^{3n}}{\zeta^{3n}} + \dots \right\}}_{B(\zeta)} \right] d\bar{\zeta} \quad (12)$$

After manipulating and integration, we can get the following result.

$$\oint_c w d\bar{w} = U^2 \oint_c \left[-\frac{a^4}{\zeta^2} d\bar{\zeta} + \underbrace{\frac{2}{n(n-1)} \frac{a^2 e^{-2i\alpha}}{\zeta^2} \frac{a^n}{\zeta^{n-1}} d\bar{\zeta} + \frac{2}{n} \frac{a^2 e^{2i\alpha}}{\zeta} \frac{a^n}{\zeta^n} d\bar{\zeta}}_{C(\zeta)} + A(\zeta) \times B(\zeta) d\bar{\zeta} \right] \quad (13)$$

The line integral $C(\zeta)$ in Eq. (13) is not equal zero for 'n=2' and it becomes zero for the arbitrary 'n' but for 'n=2'. It means that in case of the plate the virtual mass depends on the angle of attack, while in case of $n \geq 3$ it does not depend on the angle of attack. This looks strange on the face of things because of the asymmetry of the body, in which the body is not the point symmetry. However, on getting that the regular polygon tends to the 'circle' when 'n' tends to the infinity, the result seems natural. But, on the way to the limit the body shape is 'not' a point symmetry like a circle.

We can get the results of the arbitrary 'n' as shown in the next.

Case 1. n=2 (flat plate)

After manipulating Eq.(13), we can get the next result,

$$\oint_c w d\bar{w} = -2\pi i U^2 a^2 (-2 + e^{-2i\alpha} + e^{2i\alpha}) \quad (14)$$

Therefore the total kinetic energy according to the 'body disturbance' (Milne-Thomson) is given as follows.

$$T = -\frac{1}{4} i \rho \oint_c w d\bar{w} = \frac{1}{2} U^2 \underbrace{\rho \pi a^2 (2 - e^{-2i\alpha} - e^{2i\alpha})}_{M'} \quad (15)$$

where M' means the 'virtual mass'.

The virtual masses of the flat plate of $4a$ in length are shown in Table 1.

Table 1 Virtual Mass (flat plate)

α (Angle of Attack)	M' (Virtual Mass)
0	0
$\pi/4$	$\rho \times 2\pi a^2$
$\pi/2$	$\rho \times 4\pi a^2$
...	...
...	...

Case 2. $n \geq 3$

After manipulating Eq. (13), we can get the next result,

$$\oint_c w d\bar{w} = 2\pi i U^2 a^2 + U^2 \oint_c [A(\zeta) \times B(\zeta)] d\bar{\zeta} \quad (16)$$

Therefore the total kinetic energy with respect to the 'body disturbance' (Milne-Thomson) is given in Eq. (17).

$$(1) \quad T = -\frac{1}{4} i \rho \oint_c w d\bar{w} = \frac{1}{2} U^2 \rho \pi a^2 \left[1 + \underbrace{\left(\frac{2}{n} \right)^2 \frac{1}{n-1} + \left\{ \frac{\frac{2}{n}(\frac{2}{n}-1)}{2!} \right\}^2 \frac{1}{(2n-1)} + \left\{ \frac{\frac{2}{n}(\frac{2}{n}-1)(\frac{2}{n}-2)}{3!} \right\}^2 \frac{1}{(3n-1)} + \dots}_{M'} \right] \quad (17)$$

where M' means the 'virtual mass'.

The virtual mass of the regular polygon is already independent of the angle of attack, which is the noteworthy result.

Calculated examples are shown in Figure 2. Here the equilateral polygon is inscribed in a circle such that the radius is

$$a \left(1 + \frac{2}{n(n-1)} - \frac{\frac{2}{n}(\frac{2}{n}-1)}{2!(2n-1)} + \dots \right)$$

n	M' (Virtual Mass)	circumscribed circle radius
3(triangle)	$\rho \pi a^2 \left(1 + \frac{2}{3^2} + \frac{1}{3^4 \cdot 5} + \frac{2}{3^8} + \dots \right)$	$a \left(1 + \frac{1}{3} + \frac{1}{3^2 \cdot 5} + \frac{1}{3^4 \cdot 2} + \dots \right)$
4(square)	$\rho \pi a^2 \left(1 + \frac{1}{2^2 \cdot 3} + \frac{1}{2^6 \cdot 7} + \frac{1}{2^8 \cdot 11} + \dots \right)$	$a \left(1 + \frac{1}{2 \cdot 3} + \frac{1}{2^3 \cdot 7} + \frac{1}{2^4 \cdot 11} + \dots \right)$
\vdots	\vdots	\vdots
\vdots	\vdots	\vdots
∞ (circle)	$\rho \pi a^2$	a

Figure 2 Virtual Mass (regular polygon)

3.0 CONCLUSION

The exact solution of virtual mass of regular polygon with angle of attack is shown by using complex velocity potential. Although the shape is not a point symmetry at the origin, that is not dependent on the angle of attack, except the flat plate ($n=2$).

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